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Legendary Line Domination in L(L(G))

Dr. V. Mohana Selvi[1] and P. Kavitha[2]
Assistant Professor[1,2]
Department of Mathematics[1,2]
Nehru Memorial College[1,2]
Puthanampatti[1,2]- 621007
Email: vmohanaselvi@gmail.com[1] , kavitha.thamilmani@gmail.com[2]

Abstract- Harary and Norman introduced the line graph *L(G)* in the year 1960. We introduced the legendary domination number by combining the domination concept both in graph G and its line graph *L(G) or* G_l for some simple graphs. In this paper, we extend the legendary domination for line graphs G_l and its line graph $L(G_l)$ *or* G_{ll}. Also, the corresponding legendary line domination number $\gamma_{ll}(G_{ll})$ defined. Also, we studied its graph theoretical properties and obtained some bounds in terms of elements of *G*.

Keywords- Line graph; Domination number; Legendary domination number; Legendary line domination number.

AMS Subject Classification: 05C69, 05C76

1. INTRODUCTION

A graph $G = (V, E)$ we mean a finite, undirected, simple and connected graph with p vertices and q edges. Terms not defined here are used in the sense of Harary in [1]. The line graph of *G*, denoted by G_l, is a graph with vertex set *E(G)*, where vertices x and y are adjacent in G_l if and only if edges x and y in G share a common vertex. This was introduced by Harary and Norman in [2]. A set $D \subseteq V$ is a dominating set of G, if every vertex in $V\backslash D$ is adjacent to atleast one vertex in *D*. The domination number $\gamma(G)$ of *G* is the minimum cardinality of a dominating set of G. This concept was introduced by Ore in [5]. A edge (line) set $S \subseteq E$ is a edge (line) dominating set of G, if every edge (line) in E\S is adjacent to atleast one edge (line) in *S*. The domination number $\gamma'(G)$ of *G* is the minimum cardinality of a edge (line) dominating set of G. It was introduced by S. R. Jayaram in [3]. Throughout this paper, we take G is a *(p, q)* simple, connected and undirected graph with atleast three vertices. $G_l=(V_l, E_l)$ and $G_{ll} = (V_{ll}, E_{ll})$ be respectively denote the line graph of G and line graph of G_l. Also we take $V = \{u_1, u_2, \ldots, u_p\}$, $E = \{e_1, e_2, \ldots, e_q\}$, $V_l = \{v_1=e_1, v_2=e_2, \ldots, v_{ql}=e_q\}$, $E_l = \{f_1, f_2, \ldots, f_{ql}\}$, $V_{ll} = \{w_1=f_1, w_2=f_2, \ldots, w_{ql}=f_{ql}\}$ and $E_{ll} = \{g_1, g_2, \ldots, g_{ql}\}$. A vertex set $D \subseteq V_l$ is said to be a legendary dominating set of G_l if *G* and G_l both have a dominating set of cardinality *D*. The legendary domination number is the minimum cardinality taken over all legendary dominating sets of G_l and is denoted by $\gamma_l(G_l)$ which was introduced in [4]. In this paper, we extend the legendary domination number for G_l and G_{ll} and introduced the legendary line domination number of G_{ll}, it is denoted by $\gamma_{ll}(G_{ll})$. The characteristics of this parameter was studied and its exact value was found for some standard graphs like cycle, path, crown, comb, star, wheel graph, triangular snake graph and complete graph.

2. MAIN RESULTS

2.1 LEGENDARY LINE DOMINATION NUMBER

Definition 2.1.1

A dominating set *L* of the Line graph G_l is said to be legendary line dominating set (LLD-set) of G_{ll}, if G_{ll} has a dominating set of cardinality *L*. The legendary line domination number of G_{ll} is the minimum cardinality taken over all legendary line dominating sets of G_{ll} and is denoted by $\gamma_{ll}(G_{ll})$.

Example 2.1.2

For the following graph G_{ll} in Figure 2.1, the vertex set $L=\{w_2,w_4\}$ is a minimum legendary dominating set. Since in G_l, $L_1=\{v_1, v_4\}$ is a dominating set with cardinality 2. Hence $\gamma_{ll}(G_{ll})=2$.

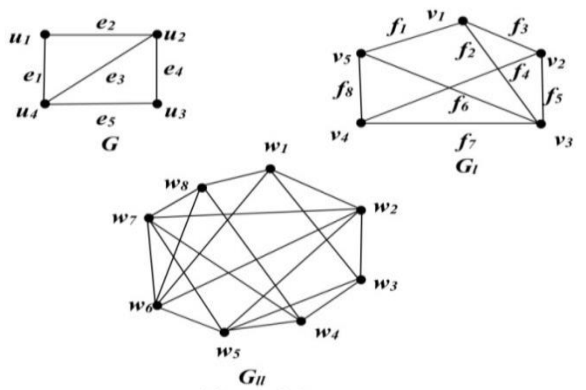

Figure 2.1

Remarks2.1.3

For the study of legendary line domination number of G_{ll}, without loss of generality, we consider all the graphs here are having atleast one LLD – set.

2.2 BOUNDS AND RESULTS ON $\gamma_{ll}(G_{ll})$

The following theorem gives the relation between domination number, legendary domination number, legendary line domination number.

Theorem 2.2.1

For any graph G, $\gamma(G)\le\gamma_l(G_l)\le\gamma_{ll}(G_{ll})$.

Proof.

Since every legendary line dominating set of G_{ll} is a Legendary dominating set of G_l, gives

$$\gamma_l(G_l)\le\gamma_{ll}(G_{ll}) \qquad \text{……... (1)}$$

Similarly, every legendary dominating set of G_l is a dominating set of G, gives

$$\gamma(G)\le\gamma_l(G_l) \qquad \text{……..... (2)}$$

The result followed from Eq. (1) and Eq. (2).

Theorem 2.2.2

For the cycle C_n, $\gamma_{ll}(G_{ll})=\left\lfloor\frac{n}{2}\right\rfloor, n\ge 3$.

Proof.

Let G be the cycle C_n with atleast three vertices. Let $V_{ll}=\{w_1,w_2,\ldots\ldots w_n\}$ the vertex set of G_{ll}.

Case (i) n is odd

In this case, the vertex set $L=\left\{w_{2i}\,/\,i=1,2,3,\ldots\ldots,\frac{n-1}{2}\right\}$ of G_{ll} is the LLD-set of G_{ll} gives.

$$\gamma_{ll}(G_{ll})\le|L|=\frac{n}{2}\le\left\lfloor\frac{n}{2}\right\rfloor \quad \text{……….(1)}$$

On the other hand, let L be a γ_{ll}-set of G_{ll} then L must contain atleast $\frac{n}{2}$ alternative vertices and hence

$$\gamma_{ll}(G_{ll})=|L|\ge\frac{n}{2}\ge\left\lfloor\frac{n}{2}\right\rfloor \text{…....(2)}$$

Hence the result is followed from Eq. (1) and Eq. (2).

Case (ii) n is even

In this case, the vertex set $L=\left\{w_{2i-1}\,/\,i=1,2,3,\ldots\ldots,\frac{n}{2}\right\}$ of G_{ll} is the LLD-set of G_{ll} gives, $\gamma_{ll}(G_{ll})\le|L|=\frac{n-1}{2}\le\left\lfloor\frac{n}{2}\right\rfloor$ ....(3)

For the other inequality, let L be a γ_{ll}-set of G_{ll} then$L$ must contain atleast $\frac{n}{2}$ alternative vertices and hence $\gamma_{ll}(G_{ll})\le|L|\ge\frac{n}{2}=\left\lfloor\frac{n}{2}\right\rfloor$ ......(4)

Hence Eq. (3) and Eq. (4) proves the result.

Example 2.2.3

For the following graph G_{ll} in Figure 2.2, the vertex set $L=\{w_1, w_3\}$ is a minimum legendary dominating set. Since in G_l, $L_1=\{v_1,v_3\}$ is a dominating set with cardinality 2. Hence $\gamma_{ll}(G_{ll})=2$.

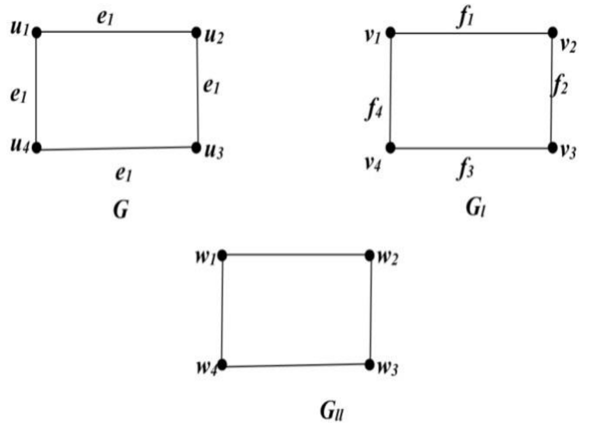

Figure 2.2

Theorem 2.2.4

For the path P_n $\gamma_{ll}(G_{ll})=\left\lfloor\frac{n-2}{2}\right\rfloor, n\ge 4$.

Proof.

Let G be the path P_n with atleast four vertices. Let $V_{ll} = \{w_1, w_2, \ldots\ldots w_{n-2}\}$ be the vertex set of G_{ll}.

Case (i) n is odd

In this case, the vertex set $L = \left\{ w_{2i} \,/\, i = 1,2,3,\ldots\ldots, \frac{n-3}{2} \right\}$ of G_{ll} is the LLD-set of G_{ll} gives,

$$\gamma_{ll}(G_{ll}) \leq |L| = \frac{n-1}{2} \leq \left\lfloor \frac{n-2}{2} \right\rfloor \quad \ldots\ldots(1)$$

On the otherhand, let L be a γ_{ll}-set of G_{ll} then L must contain atleast $\frac{n-2}{2}$ alternative vertices and hence $\gamma_{ll}(G_{ll}) = |L| \geq \frac{n-2}{2} \geq \left\lfloor \frac{n-2}{2} \right\rfloor \quad \ldots.(2)$

The result is followed from Eq. (1) and Eq. (2).

Case (ii) n is even

In this case, the vertex set $L = \left\{ w_{2i-1} \,/\, i = 1,2,3,\ldots\ldots, \frac{n-2}{2} \right\}$ of G_{ll} is the LLD-set of G_{ll} gives,

$$\gamma_{ll}(G_{ll}) \leq |L| = \frac{n-2}{2} = \left\lfloor \frac{n-2}{2} \right\rfloor \quad \ldots.(3)$$

For the other inequality, let L be a γ_{ll}-set of G_{ll} then L must contain atleast $\frac{n-2}{2}$ alternative vertices and hence $\gamma_{ll}(G_{ll}) = |L| \leq \frac{n-2}{2} = \left\lfloor \frac{n-2}{2} \right\rfloor \quad \ldots..(4)$

Hence Eq. (3) and Eq. (4) proves the result.

Example 2.2.5

For following graph G_{ll} in Figure 2.3, the vertex set $L=\{w_1, w_4\}$ is a minimum legendary dominating set. Since in G_l, $L_1=\{v_1, v_4\}$ is a dominating set with cardinality 2. Hence $\gamma_{ll}(G_{ll}) = 2$.

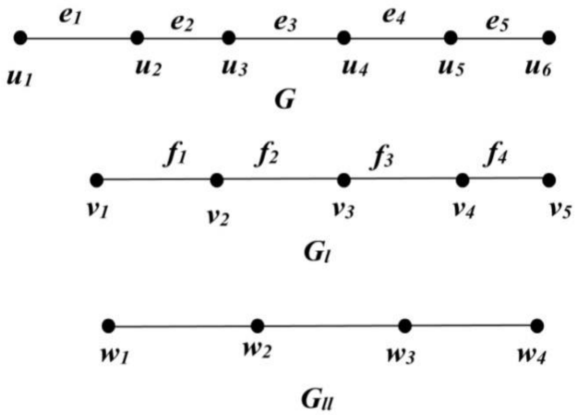

Figure 2.3

Theorem 2.2.6

For the crown graph C_n^+ $\gamma_{ll}(G_{ll}) = n,\ n \geq 3$.

Proof.

Let G be the crown graph C_n^+ with atleast six vertices. Let $V_{ll}=\{w_1, w_2, \ldots\ldots, w_{3n}\}$ be the vertex set of G_{ll}.

The vertex set $L = \{w_{3i-2} \,/\, i = 1,2,3,\ldots\ldots, n\}$ of G_{ll} is the LLD-set of G_{ll} gives,

$$\gamma_{ll}(G_{ll}) \leq |L| = n \quad \ldots\ldots..(1)$$

For the other hand, let L be a γ_{ll}-set of G_{ll} then L must contain atleast n alternative vertices and hence

$$\gamma_{ll}(G_{ll}) \leq |L| = n \quad \ldots\ldots...(2)$$

The result is followed from Eq. (1) and Eq. (2).

Example 2.2.7

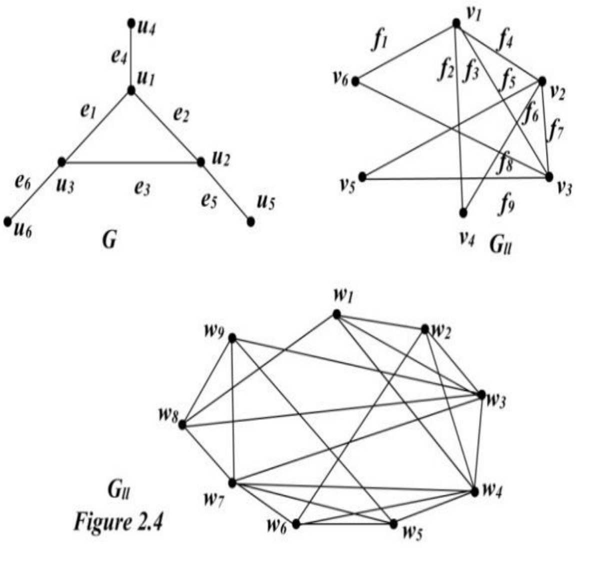

Figure 2.4

For graph G_{ll} in Figure 2.4, the vertex set $L=\{w_1, w_4, w_7\}$ is a minimum legendary dominating set. Since in G_l, $L_1=\{v_1, v_3, v_5\}$ is a dominating set with cardinality 3. Hence $\gamma_{ll}(G_{ll}) = 3$.

Theorem 2.2.8

For the comb graph P_n^+, $\gamma_{ll}(G_{ll}) = n-2,\ n \geq 3$.

Proof.

Let G be the comb graph P_n^+ with atleast six vertices. Let $V_{ll}=\{w_1,w_2,...,w_{3n-4}\}$ be the vertex set of G_{ll}. The vertex set $L=\{w_{3i}/i=1,2,....,(n-2)\}$ of G_{ll} is the LLD-set of G_{ll} gives,

$\gamma_{ll}(G_{ll}) \leq |L| = n-2$ ....(1)

On the other hand, let L be a γ_{ll}-set of G_{ll} then L must contain atleast n-2 alternative vertices and hence

$\gamma_{ll}(G_{ll}) = |L| \geq n-2$ ....(2)

The result is followed from Eq. (1) and Eq. (2).

Example 2.2.9

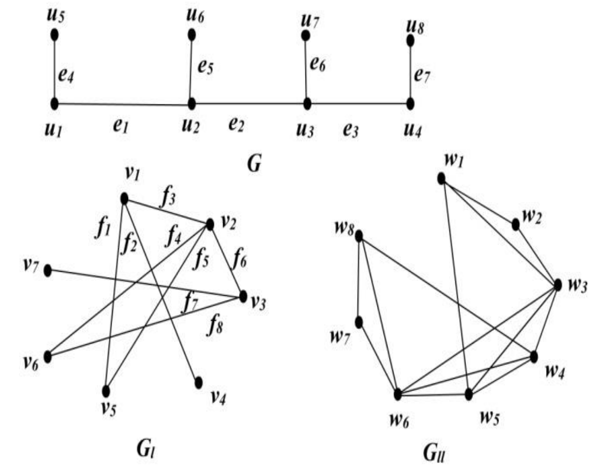

Figure 2.5

For graph G_{ll} in Figure 2.5, the vertex set $L=\{w_3,w_6\}$ is a minimum legendary dominating set. Since in G_l, $L_1=\{v_1, v_3\}$ is a dominating set with cardinality 2. Hence $\gamma_{ll}(G_{ll}) = 2.$

Theorem 2.2.10

For the triangular snake graph nC_3, $\gamma_{ll}(G_{ll}) = |L| \geq n-2$.

Proof.

Let G be the triangular snake graph nC_3 with atleast five vertices. Let $V_{ll}=\{w_1,w_2,\ldots\ldots w_{7n-4}\}$ be the vertex set of G_{ll}. The vertex set $L=\{w_{7i-2} / i=1,2,3,......,n-1\} \cup \{w_{7n-4}\}$ of G_{ll} is the LLD-set of G_{ll} gives,

$\gamma_{ll}(G_{ll}) \leq |L| = n$ ....(1)

On the other hand, let L be a γ_{ll}-set of G_{ll} then L must contain atleastn alternative vertices and hence

$\gamma_{ll}(G_{ll}) = |L| \geq n$ ....(2)

The result is followed from Eq. (1) and Eq. (2).

Example 2.2.11

For the following graph G_{ll} in Figure 2.6, the vertex set $L=\{w_5,w_9\}$ is a minimum legendary dominating set. Since in G_l, $L_1=\{v_2, v_5\}$ is a dominating set with cardinality 2. Hence $\gamma_{ll}(G_{ll}) = 2.$

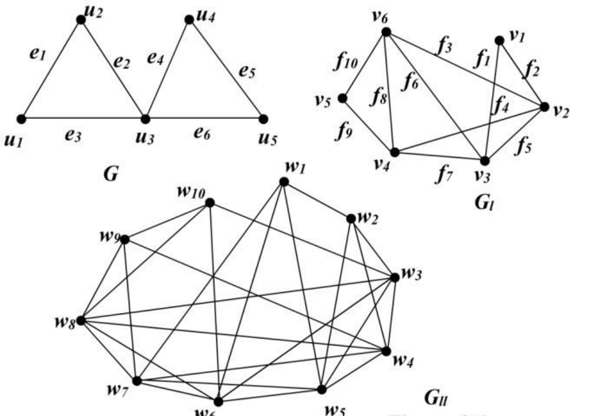

Figure 2.6

Theorem 2.2.12

For the wheel graph W_n, $\gamma_{ll}(G_{ll}) = n,\ n \geq 3$.

Proof.

Let G be the wheel graph W_n with at least four vertices. Let $V_{ll} = \{w_1,w_2,...,w_{6n-6}\}$ be the vertex set of G_{ll}.

The vertex set $L=\{w_{3i}/i=1,2,....,n\}$ of G_{ll} is the LLD-set of G_{ll} gives,

$\gamma_{ll}(G_{ll}) \leq |L| = n$ ....(1)

On the other hand, let L be a γ_{ll}-set of G_{ll} then L must contain atleast n alternative vertices and hence

$\gamma_{ll}(G_{ll}) = |L| \geq n$ ....(2)

The result is followed from Eq. (1) and Eq. (2).

Example 2.2.13

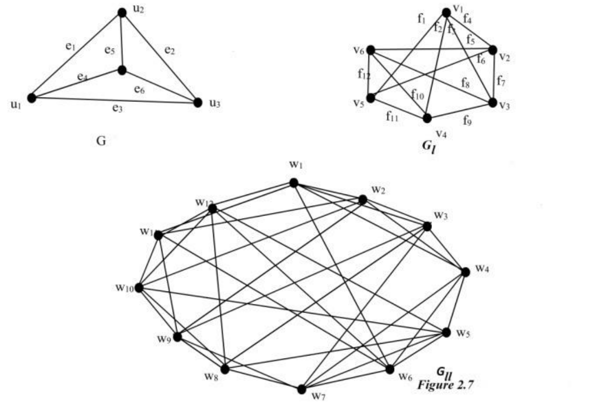

Figure 2.7

For graph G_{ll} in Figure 2.7, the vertex set $L=\{w_3,w_6,w_9\}$ is a minimum legendary dominating set. Since in G_l, $L_1=\{v_1,v_3,v_5\}$ is a dominating set with cardinality 3. Hence $\gamma_{ll}(G_{ll})=3$.

Theorem 2.2.14

For the $W_n^{\,+}$ graph, $\gamma_{ll}(G_{ll})=n+1,\ n\geq 3$.

Proof.

Let G be the $W_n^{\,+}$ graph with atleast seven vertices and $\{w_1,w_2,\ldots\ldots w_{3(n-2)}\}$ be the vertex set of G_{ll}.

The vertex set $L=\{w_{5i+1}/i=1,2,3,\ldots\ldots,n\}$ of G_{ll} is the LLD-set of G_{ll} gives,

$\gamma_{ll}(G_{ll})\leq|L|=n+1$ ....(1)

On the other hand, let L be a γ_{ll}-set of G_{ll} then L must contain atleast n+1 alternative vertices and hence

$\gamma_{ll}(G_{ll})=|L|\geq n+1$ ....(2)

The result is followed from Eq. (1) and Eq. (2).

Example 2.2.15

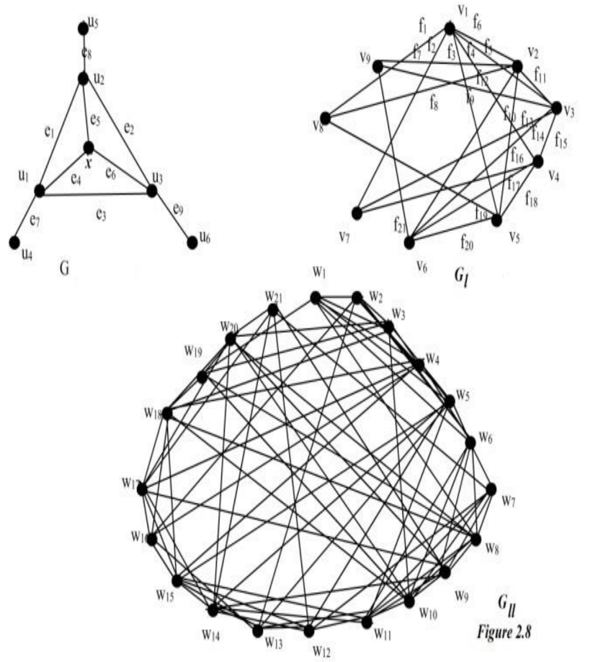

Figure 2.8

For graph G_{ll} in Figure 2.8, the vertex set $L=\{w_6,w_{11},w_{16},w_{21}\}$ is a minimum legendary dominating set. Since in G_l, $L_1=\{v_1\ ,v_3,v_5,v_7\}$ is a dominating set with cardinality 4. Hence $\gamma_{ll}(G_{ll})=4$.

Theorem 2.2.16

For the star graph $K_{1,n}$, $\gamma_{ll}(G_{ll})=\left\lfloor \frac{n}{2}\right\rfloor,\ n\geq 2$.

Proof.

Let G be the star graph $K_{1,n}$ with atleast three vertices. $V_{ll}=\{w_1,w_2,\ldots,w_{n(n-1)/2}\}$ be the vertex set of G_{ll}.

Case (i) n is odd

In this case, the vertex set $L=\{w_{(n-1)i}/i=1,2,\ldots\ldots,(n-1)/2\}$ of G_{ll} is the LLD-set of G_{ll} gives,

$$\gamma_{ll}(G_{ll})\leq|L|=\frac{n-1}{2}\leq\left\lfloor\frac{n}{2}\right\rfloor \text{........(1)}$$

On the otherhand, let L be a γ_{ll}-set of G_{ll} then L must contain atleast (n-1)/2 alternative vertices and hence

$\gamma_{ll}(G_{ll})\leq|L|\geq\frac{n}{2}\geq\left\lfloor\frac{n}{2}\right\rfloor$ ....(2)

Now, the result is followed from Eq. (1) and Eq. (2).

Case (ii) n is even

In this case, the vertex set $L=\{w_{(n-1)i}/i=1,2,\ldots,n/2\}$ of G_{ll} is the LLD-set of G_{ll} gives,

$\gamma_{ll}(G_{ll})\leq|L|=\frac{n-1}{2}\leq\left\lfloor\frac{n}{2}\right\rfloor$ ....(3)

For the other inequality, let L be a γ_{ll}-set of G_{ll} then L must contain atleast n/2 alternative vertices and hence

$$\gamma_{ll}(G_{ll}) = |L| \geq \frac{n}{2} = \left\lfloor \frac{n}{2} \right\rfloor \qquad ....(4)$$

Hence Eq. (3) and Eq. (4) proves the result.

Example 2.2.17

For the following graph G_{ll} in Figure 2.9, the vertex set $L=\{w_3, w_6\}$ is a minimum legendary dominating set. Since in G_l, $L_1=\{v_3, v_6\}$ is a dominating set with cardinality 2. Hence $\gamma_{ll}(G_{ll}) = 2.$

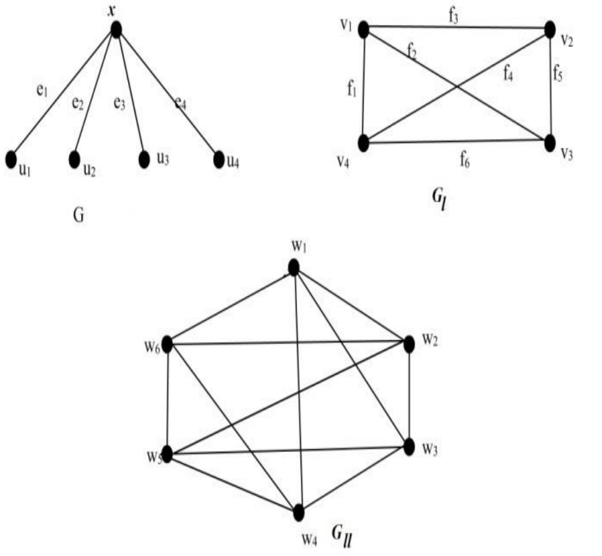

Figure 2.9

Theorem 2.2.18

For the complete graph K_n,

$$\gamma_{ll}(G_{ll}) = \begin{Bmatrix} 1, & n=3 \\ 3, & n=4 \\ 2n-5, & n \geq 5 \end{Bmatrix}.$$

Proof.

Let G be the complete graph K_n with atleast three vertices and $\left\{ w_1, w_2, ..., w_{\frac{n(n-1)(n-2)}{2}} \right\}$ be the vertex set of G_{ll}.

Case (i) n =3

In this case, the vertex set $L=\{w_1\}$ of G_{ll} is the LLD-set of G_{ll} gives,

$$\gamma_{ll}(G_{ll}) = 1.$$

Case (ii) n =4

In this case, the vertex set $L=\{w_4, w_8, w_{12}\}$ of G_{ll} is the LLD-set of G_{ll} gives,

$$\gamma_{ll}(G_{ll}) \leq |L| = 3 \qquad ....(1)$$

On the otherhand, let L be a γ_{ll}-set of G_{ll} then L must contain atleast three alternative vertices and hence

$$\gamma_{ll}(G_{ll}) = |L| \geq 3 \qquad ....(2)$$

Now, the result is followed from Eq. (1) and Eq. (2).

Case (iii) n ≥ 5

In this case, the vertex set $L=\{w_{(2n-4)i}/i=1,2,3..........,n\}$ of G_{ll} is the LLD-set of G_{ll} gives,

$$\gamma_{ll}(G_{ll}) \leq |L| = 2n-5 \qquad ....(3)$$

For the other inequality, let L be a γ_{ll}-set of G then L must contain atleast *2n-5* alternative vertices and hence

$$\gamma_{ll}(G_{ll}) = |L| \geq 2n-5 \qquad ....(4)$$

Hence Eq. (3) and Eq. (4) proves the result.

Example 2.2.19

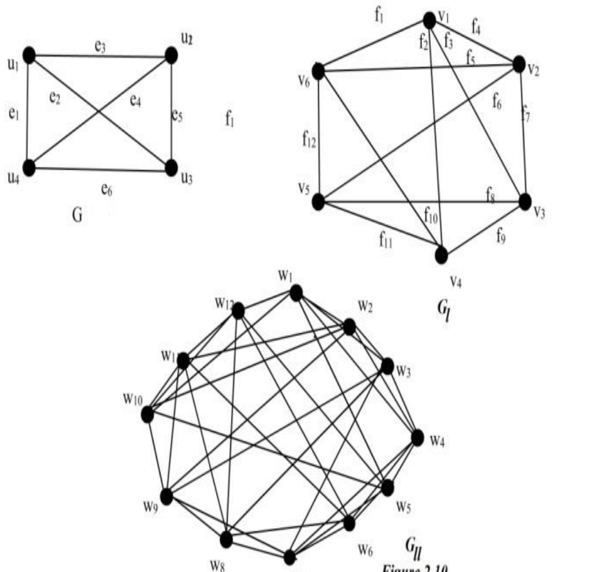

Figure 2.10

For graph G_{ll} in Figure 2.10, the vertex set $L=\{w_4,w_8,w_{12}\}$ is a minimum legendary dominating set. Since in G_l, $L_1=\{v_1, v_3, v_5\}$ is a dominating set with cardinality 2. Hence $\gamma_{ll}(G_{ll}) = 2.$

The following theorem gives the necessary and sufficient conditions for a vertex set to be a legendary line dominating set of G_{ll}.

Theorem 2.2.20

A vertex dominating set L of a graph G_{ll} is a legendary line dominating set of G_{ll} if and only if there exists a vertex v in L such that $N(v) \subseteq L$.

Proof.

Assume L is a legendary line dominating set of G_{ll} and there is no vertex v in L satisfying $N(v) \subseteq L$.

Now, for any $v \in L$, each of its adjacent vertices belongs to *V\L*. Thus, *V\L* becomes a vertex dominating set, which is a contradiction to our assumption that L is a legendary line dominating set of G_{ll}. Hence, $N(v) \subseteq L$ for some v $\in$ L.

Conversely, suppose there exists vertex *v* in *L* such that $N(v) \subseteq L$ then *v* cannot be dominated by any vertex of *V\L*, it shows that *V\L* is not a vertex dominating set of G_{ll} and hence L is a legendary line dominating set of G_{ll}.

Theorem 2.2.21

For any graph *G*, $\gamma_{ll}(G_{ll}) \leq \gamma(G_{ll}) + \delta(G_{ll}) + 1.$

Proof.

Let *L* be a γ_{ll} -set of G_{ll} and v be a vertex of G_{ll} such that $\deg(v) = \delta(G_{ll})$. Then either v $\subseteq$ L or some vertex w adjacent to v belongs to *L*. Let L' = L $\cup$ N[v]. Since *L* is γ_{ll} -set of G_{ll}, *L'* is also a vertex dominating set of G_{ll}. As $L \subseteq N(v)$ by Theorem 2.2.20, *L'* is a legendary line dominating set of G_{ll}.

Hence, $\gamma_{ll}(G_{ll}) \leq |L \cup N[v]| \leq |L| + |N[v]|$

That is, $\gamma_{ll}(G_{ll}) \leq \gamma(G_{ll}) + \delta(G_{ll}) + 1$.

3. CONCLUSION

In this paper, we found the exact values of legendary line domination number of G_{ll} for cycle, path, crown, comb, triangular snake graph, star graph and wheel graph.

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